On the Symmetry Approach to Reduction of Partial Differential Equations

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Abstract

We propose the symmetry reduction method of partial differential equations to the system of differential equations with fewer number of independent variables. We also obtain generalized sufficient conditions for the solution found by conditional symmetry method to be an invariant one in classical sense.

1.INTRODUCTION

In recent years the symmetry method is often used for reduction of partial differential equations to the equations with fewer number of independent variables and thus for construction of exact solutions for different mathematical physics problems. To construct a corresponding ansatz generators of classical Lie point transformations are used as well as operators of conditional symmetry. In this connection the application of combination of conditional and generalized symmetry is fruitful as was shown in [1, 2] on the examples of evolution equation in two-dimensional case (see also [3]). In [4] Svirshchevskii proposed the symmetry reduction method based on the invariance of linear ordinary differential equations (see also [5]). It is the symmetry explanation of "nonlinear" separation of variables [6] for the evolution type equations.

Here we propose an approach applicable for symmetry reduction of partial differential equations which are not restricted to evolution type ones. It can be used in multi-dimensional case. This approach is the generalization of the method introduced in [4].

1 Generalized symmetry and reduction of partial differential equations

Let us consider partial differential equation

$$U(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}) = 0, \tag{1}$$

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where $x = (x_1, x_2, \ldots, x_n)$, $u = u(x) \in C^k(\mathbb{R}^n, \mathbb{R}^1)$, and $\underset{k}{u}$ denotes all partial derivatives of k-th order. Replacing u by $u + \epsilon w$ and then equating coefficients at ϵ in Taylor series expansion we obtain linearized equation (1)

$$L(x, u, w) = 0. \tag{2}$$

It has been proved that the following property is fulfilled in this case. If equation (1) admits Lie–Bäcklund vector field $Q = \eta(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{r}{u})\partial_u$ and $u = f(x)$ is a solution of equation (1) then

$$w = Qu|_{u=f(x)} \tag{3}$$

is a solution of equation (2).

This property is illustrated by the connection between the solutions of Liouville and Moutard equations. It is well known that the Liouville equation

$$u_{xy} = 2 \exp u \tag{4}$$

is invariant with respect to the Lie group of transformations with generator

$$Q_1 = f(x)\partial_x + g(y)\partial_y - (f' + g')\partial_u,$$

where $f(x)$ and $g(y)$ are arbitrary smooth functions. Then from the solution

$$u = \ln \frac{X'Y'}{(X+Y)^2}$$

of Liouville equation we easily obtain the solution

$$u = \frac{X_1'}{X'} + \frac{Y_1'}{Y'} - 2\frac{X_1 + Y_1}{X + Y}$$

of the Moutard equation

$$w_{xy} = 2\frac{X'Y'}{(X+Y)^2}w$$

with potential $V = 2\frac{X'Y'}{(X+Y)^2}$, where $X(x)$, $Y(y)$ are arbitrary smooth functions of their arguments, $fX' = X_1$, $gY' = Y_1$, by using (3).

This property can also be used for reduction of partial differential equations to system of equations with smaller number of independent variables. For simplicity consider ordinary differential equation

$$H(x, u, \ldots, \underset{m'}{u}) = 0, \tag{5}$$

where $\underset{m'}{u}$ denotes the derivative of u with respect to one variable x_1 of m'-th order, and H are a smooth function of its arguments. Suppose that Q is the operator of Lie–Bäcklund symmetry of equation (5). Let

$$u = F(x, C_1, \ldots, C_{m'}), \tag{6}$$

where F is a smooth function of variables $x, C_1, \ldots, C_{m'}$, $C_1, \ldots, C_{m'}$ are arbitrary functions of parametric variables $x_2, x_3, \ldots, x_n$, be a general solution of equation (5). Then operator Q transforms solution (6) to the solution of linearized version of equation (5) i.e. the linear homogeneous ordinary differential equation. It means that Q maps the set of solutions (6) into a m'–dimensional vector space M. Moreover, we proved that partial derivatives $\frac{\partial F}{\partial C_i}$, $i = \overline{1, m'}$ form the basis of M provided that H and F are sufficiently smooth functions of their arguments. In this connection the following statement holds.

Theorem 1 *Let equation (5) be invariant with respect to the Lie–Bäcklund operator Q. Then the ansatz*

$$u = F(x, \phi_1, \phi_2, \ldots, \phi_{m'}), \tag{7}$$

where $\phi_1, \phi_2, \ldots, \phi_{m'}$ depend on $n-1$ variables $x_2, x_3, \ldots, x_n$ reduces partial differential equation

$$\eta(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{r}{u}) = 0 \tag{8}$$

to the system of k_1 equations for unknown functions $\phi_1, \phi_2, \ldots, \phi_m$ with $n-1$ independent variables and $k_1 \le m'$.

Taking into account the above–mentioned arguments we proved the theorem. It can be easily generalize for $\phi_1, \phi_2, \ldots, \phi_{m'}$ depending on $\omega_l(x)$, $l = \overline{1, n-1}$, where $\omega_l(x)$ are some functions of variables x. Note that the case when equation (5) is linear ordinary differential equation and η is the function of special type (evolution type) was considered in [4].

We consider several examples illustrating the application of the theorem. Firstly consider equation

$$u_t = f(u_x)u_{xx}, \tag{9}$$

where $f(u_x)$ is a smooth function of u_x. Let us consider the equation (5) in the following form

$$u_{xx} = u_x^3. \tag{10}$$

In accordance with our approach we study the invariance of the equation (10) with respect to Lie–Bäcklund operator

$$K_1 = (u_t - f(u_x)u_{xx})\partial_u. \tag{11}$$

Function $f(u_x)$ is determined from the condition of invariance of the equation (10) with respect to the operator (11).It has the form

$$f = \frac{A}{u_x^3} + \frac{B}{u_x^2},$$

where A, B are arbitrary real constants. Then we proved that the equation (10) admits the operators

$$K_2 = uu_x\partial_u, \qquad K_3 = h(u + u_x^{-1})\partial_u,$$

where h is arbitrary smooth function. Therefore the equation (10) is invariant with respect to the group of Lie–Bäcklund transformations with infinitesimal operator $\alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3$, where α_1, α_2, α_3 are arbitrary real constants. From this it follows that the theorem can be used for the equation

$$u_t = (\frac{A}{u_x^3} + \frac{B}{u_x^2})u_{xx} + \lambda uu_x + \lambda_1 h(u + u_x^{-1}), \tag{12}$$

where λ, λ_1 are arbitrary real constants. The ansatz corresponding to the equation (10) has the form

$$u = \phi_2(t) - (\phi_1(t) - 2x)^{1/2}, \tag{13}$$

where $\phi_1(t)$ and $\phi_2(t)$ are unknown functions. The ansatz (13) reduces the equation (12) to the system of ordinary differential equations

$$\phi_2' = A - \lambda + \lambda_1 h(\phi_2), \qquad -\phi_1' = 2(B + \lambda\phi_2). \tag{14}$$

For different $h(\phi_2)$ we receive different solutions. Let $h(\phi_2) = \phi_2$. In this case the solution of the system (14) is

$$\phi_2 = Ce^{\lambda_1 t} - (A - \lambda)\lambda_1^{-1}, \quad -\phi_1 = 2(B - (A - \lambda)\lambda\lambda_1^{-1})t + 2\lambda_1^{-1}\lambda Ce^{\lambda_1 t} - C_1.$$

Substituting this solution into (13) we obtain the solution

$$u = Ce^{\lambda_1 t} - (A - \lambda)\lambda_1^{-1} - [2((A - \lambda)\lambda\lambda_1^{-1} - B)t - 2\lambda_1^{-1}\lambda Ce^{\lambda_1 t} + C_1 - 2x]^{1/2}$$

of equation (12). This solution can not be obtain by using classical Lie method of point symmetry.

Note that the ansatz (13) can be used for reduction of equations obtained by commutating of operators K_1, K_2, K_3 also.

Next we show the application of the method to equation associated with inverse scattering problem. As we know the group–theoretical background of the inverse scattering problem method was given for the first time in [7].

Namely we study the symmetry of the linear ordinary differential equation

$$u_{xx} = f(t, x)u. \tag{15}$$

where variable t play the role of parameter in this equation, with respect to the Lie–Bäcklund operator

$$Q_1 = (u_t + u_{xxx} - 3\frac{u_{xx}u_x}{u} + \alpha(t)u)\partial_u, \tag{16}$$

where $\alpha(t)$ is a function of t.

We have proved that the equation (15) admits operator (16) if and only if f satisfies the Korteveg-de Vrise equation in the form

$$f_t + f_{xxx} - 6ff_x = 0. \tag{17}$$

This statement is valid for arbitrary smooth $\alpha(t)$. Thus if one can construct the general solution of equation (15) for some solution $f(t, x)$ of (17) then the corresponding ansatz will reduce partial differential equation

$$u_t + u_{xxx} - 3\frac{u_{xx}u_x}{u} + \alpha(t)u = 0 \tag{18}$$

to the system of two ordinary differential equations with independent variable t. We stress that solution $f(t, x)$ should not necessarily vanish at the infinity.

To solve the Cauchy problem

$$u \,|_{t=t_0} = g(x) \tag{19}$$

for equation (18) one should construct solution $f = p(t, x)$ of equation (17) satisfying the condition $p(t_0, x) = \frac{g_{xx}}{g}$ and then to integrate ordinary differential equation

$$u_{xx} = p(t, x)u.$$

If we can construct the ansatz in this way then we reduce the Cauchy problem (19) for the equation (18) to the Cauchy problem for system of two ordinary differential equations.

In addition note that theorem 1 helped us to generalize theorem proved in [8] and concerning the sufficient conditions for the solution obtained by using conditional symmetry operators to be an invariant solution in the classical sense. Namely consider involutive family of operators

$$Q_a = \xi_{aj}(x, u)\partial_{x_j} + \eta_a(x, u)\partial_u, \qquad a = \overline{1, m}. \tag{20}$$

We have summation on repeated indeces.Suppose that the equation (1) is conditionally invariant with respect to involutive family of operators (20) and corresponding ansatz reduces this equation to ordinary differential equation of k_1-th order. Then the following statement holds.

Theorem 2 *Let equation (1) be invariant with respect to s–dimensional Lie algebra AG_s and conditionally invariant with respect to involutive family of operators $\{Q_i\}$. If the system*

$$\xi_i^l \frac{\partial u}{\partial x_l} = \eta_i(x, u)$$

is also invariant under the algebra AG_s and $s \geq k_1 + 1$, then conditionally invariant solution of equation (1) with respect to involutive family of operators $\{Q_i\}$ is an invariant solution in the classical Lie sense.

It is necessary to note that this theorem can be generalize to the case when the algebra AG_m contains Lie–Bäcklund operators too.

2 Conclusion

We showed that the symmetry of ordinary differential equations can be used for reduction of partial differential equations to the system with fewer number of independent variables. It follows from Theorem 1 that the linearity is not the necessary condition for this reduction. It is obvious that the suggested method is applicable in many-dimensional case.

In addition note that this approach can be used in solving the problem of integrability of partial differential equations. It gives the possibility to reduce this problem to the problem of integrability for ordinary differential equation. We consider this method to be important in studding quasi-exactly solvable systems.

We also obtained the generalized sufficient condition for the solution constructed by using conditional symmetry to be an invariant one in classical sense.

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